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Spectral analysis of viscous static compressible fluid equilibria

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Abstract

It is generally assumed that the study of the spectrum of the linearized Navier–Stokes equations around a static state will provide information about the stability of the equilibrium. This is obvious for inviscid barotropic compressible fluids by the self-adjoint character of the relevant operator, and rather easy for viscous incompressible fluids by the compact character of the resolvent. The viscous compressible linearized system, both for periodic and homogeneous Dirichlet boundary problems, satisfies neither condition, but it does turn out to be the generator of an immediately continuous, almost stable semigroup, which justifies the analysis of the spectrum as predictive of the initial behaviour of the flow. As for the spectrum itself, except for a unique negative finite accumulation point, it is formed by eigenvalues with negative real part, and nonreal eigenvalues are confined to a certain bounded subset of complex numbers.

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1. Introduction

It is widely believed that the initial evolution of a system governed by a nonlinear differential equation

$$\frac{\mathrm{d}w}{\mathrm{d}t} = A(w) \tag{1}$$

for initial conditions $w(0) = w_0 + x_0$ near an equilibrium state w_0 may be approximated for small t and x_0 by $w(t) \sim w_0 + x(t)$, where $x(t)$ is the solution of the linearized system

$$\begin{aligned} &\frac{\mathrm{d}x}{\mathrm{d}t} = A'(w_0)x \\ &x(0) = x_0. \end{aligned} \tag{2}$$

This linear system may still be too complex to find general solutions, so one studies simpler problems such as instability: will some solutions grow exponentially in time? This question has an immediate spectral sound to it, so it is confidently assumed that if all the spectral points of $A'(w_0)$ have negative real part, the system will be stable.

There are too many implicit assumptions here. To begin with: (a) for the true solutions to be near the linear ones for small t, the nonlinear semigroup $w_0 \to w(t)$ must be well defined and differentiable for small t; (b) for the linear evolution to make sense, $A'(w_0)$ must be the generator of a semigroup $t \to T(t)$, preferably strongly continuous (also called a C_0-semigroup), which is by no means obvious.

(c) In addition, to know the spectrum of $A'(w_0)$ it is not enough to study the spectrum of $T(t)$, which is all one really would like to need. It is true that

$$e^{t\sigma(A'(w_0))} \subset \sigma(T(t)) - \{0\} \tag{3}$$

but the equality does not hold in general: there could be spectral points in $T(t)$ not related to any of $A'(w_0)$. The inclusion above is an identity for the point spectrum (and therefore for compact $T(t)$ there are no problems) and, of course, both sets coincide if $A'(w_0)$ is self-adjoint (or normal). When neither occurs, there is still some hope: if $T(t)$ is *eventually norm-continuous* (the mapping $t \to T(t)$ is continuous from $[t_0, \infty)$ to $\mathcal{L}(E)$ for some $t_0 > 0$ see [1, p 112]), the spectral identity (3) holds.

(d) Even in the case that certain eigensolutions behave in time as $e^{\lambda t}$ does not give enough information for other initial conditions. When $A'(w_0)$ is self-adjoint, or normal, this information is provided with every desired precision by the spectral decomposition theorem. If not, the best alternative occurs when the semigroup is *almost-stable* [2,3]. This means that:

(1) The norm of $T(t)$ in $\mathcal{L}(E)$ is bounded in $[0, T]$ by a constant depending only on T.
(2) For a certain dense set in the dual space $G \subset E'$, the mapping $t \to \langle T(t)v, \phi\rangle$ is continuous in $[0, \infty)$ for any $v \in E$, $\phi \in G$.
(3) There exists a value $\rho \in (0, 1)$ such that the spectrum of $T(1)$ does not cut the circle $|z| = \rho$, and there is a finite number of eigenvalues of finite multiplicity in $|z| > \rho$.

Condition (1) is automatic for strongly continuous semigroups, and (2) is usually pretty easy to prove: one takes the domain of the transposed operator $D(A'(w_0)^t)$, or any dense space of smooth functions, for G. Condition (3), however, needs to be addressed. It holds, for instance, if $\sigma(T(1))$ is contained in a ball centred at 0 of radius less than 1.

If the semigroup is almost-stable, for any ρ such that the spectrum $\sigma(T(1))$ does not cut $|z| = \rho$, there exists a decomposition of the underlying function space E in two subspaces E_+, E_-, invariant by $T(t)$, and norms $\| \, \|_\pm$ in $E_\pm$, such that for certain positive $\varepsilon_\pm$

$$\begin{aligned} &\|T(t)|_{E_-}\| \leqslant (\rho - \varepsilon_-)^t \\ &\|T(t)^{-1}|_{E_+}\| \leqslant (\rho + \varepsilon_+)^{-t}. \end{aligned} \tag{4}$$

The norm $\|u\|' = \|u_+ + u_-\|' = \sup(\|u_+\|, \|u_-\|)$ is equivalent to the original norm in E. This decomposition of the space in solutions (roughly) decreasing like ρ^t and growing like ρ^{-t} is the best one can do to mimic the spectral theorem. In particular, if the spectrum of $T(1)$ is contained in a ball centred at 0 of radius less than 1, every solution will be exponentially decreasing in time.

In fact the study of the growth of solutions of evolutionary equations associated to non-normal operators has a long history. It has been readily recognized that for such operators the norm of the resolvent $\|(A - z)^{-1}\|$ could be small for points z far from the spectrum, exhibiting the so-called *pseudoresonance* (see [4] and references therein). That the spectrum of the operator A does not determine the size of the semigroup operator e^{tA} for t large is well known [5] and alternatives have been found [4]. Note, however, that property (d) is finer than a mere study of the growth of $\|T(t)\|$, because it deals with the growth of $\|T(t)\boldsymbol{u}_0\|$ for generic initial conditions $\boldsymbol{u}_0$.

Let us turn to concrete examples. Newtonian fluids are governed by the Navier–Stokes equations, and if they are conducting and a magnetic field is present, by the magnetohydrodynamics (MHD) equations. All fluids are more or less viscous and all plasmas posses some resistivity, but if one is justified in neglecting viscosity one obtains the inviscid or Euler equations. Their linearization around a static equilibrium state is called the linear acoustics system, and as we will see, it is associated to a normal $A'(w_0)$ (in fact $\mathrm{i}A'(w_0)$ is self-adjoint for a certain internal product). Hence problems (b)–(d) are solved in a stroke and one may study deeper questions, such as scattering [6, pp 129–142]. Viscous, but incompressible fluids are also good: the semigroup is compact and hence almost-stable, and the semigroup is differentiable, so (a)–(d) hold [7]. MHD equilibria are much more complicated, and it is difficult to find some without certain symmetries [8, 9]. If both viscosity and resistivity are omitted (ideal MHD), (b) is true but not trivial [10]: the system may be cast in a second-order self-adjoint form [11], but the spectrum is enlarged with irrelevant points. Still, there is a sizable literature on the subject, motivated by its applications to the theory of magnetic plasma confinement. If the plasma is viscous, resistive and incompressible, properties (a)–(d) essentially hold [7], because the nonlinear semigroup (defined only up to some time for dimension three) is again compact and differentiable.

Viscous, compressible fluids get the worst of both worlds: on the one hand, the viscosity term for the velocity makes the semigroup generator non self-adjoint; on the other, the lack of a dissipative term in the continuity equation makes the resolvent noncompact. We intend to show here that nonetheless properties (b)–(d) still hold, and obtain additional information on the spectrum. It will be found that its spatial distribution for the periodic case, which is a rather easy victim of the Fourier transform, remains essentially similar for the Dirichlet problem: the spectrum of the semigroup generator lies in some half-plane $\mathrm{Re}\, z < -\delta < 0$; there is only a finite accumulation point, plus $-\infty$; all the remaining spectral points are eigenvalues of finite multiplicity, and the nonreal ones are confined to a bounded subset of $\mathbb{C}$. Condition (a) depends on the existence and properties of smooth solutions of the nonlinear system up to some time and will not be addressed here.

A more difficult question would be the study of nonstatic equilibria, an important and much-studied topic [12]. Naturally these equilibria require the presence of a forcing term.

Finally, let us note that all norms throughout the paper will be L^2-ones and we will drop any subscripts on them for ease of notation.

2. The mathematical setting

The Navier–Stokes equations for a compressible barotropic fluid without external forcing are

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\mathrm{div}(\rho \boldsymbol{u}) \\ \rho \frac{\partial \boldsymbol{u}}{\partial t} &= \nu \Delta \boldsymbol{u} + \xi \nabla(\mathrm{div} \boldsymbol{u}) - \nabla p - \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} \end{aligned} \tag{5}$$

where $\boldsymbol{u}$ is the fluid velocity, ρ the density, ν the viscosity, ξ another constant with $\nu + \xi > 0$, and p the kinetic pressure. For simplicity we will assume $\xi = 0$; this only simplifies somewhat the expressions and has no bearing on the behaviour of the operator, which is dominated by the Laplacian. The relation between p and ρ is given by an independent state equation. As stated, we will assume that we are dealing with a barotropic fluid, i.e. $p = p(\rho)$, where p is a strictly increasing function of ρ: for many physical cases, p is a power of ρ, $p = C\rho^\gamma$. Boundary conditions must be added depending on the situation we are considering. A static equilibrium is given by $\boldsymbol{u} = \mathbf{0}$, $\partial/\partial t = 0$; the continuity equation vanishes identically, and the momentum one yields $\nabla p = \mathbf{0}$, which in a connected domain Ω means constant pressure, i.e. constant density. To avoid the trailing of constants, we will take this constant density as 1, and set $b = p'(1) > 0$. Now let ρ, $\boldsymbol{u}$ denote not the original magnitudes, but small perturbations of them. Note that while the density is positive, ρ may be positive or negative (although always less than 1), but since the total mass is not allowed to change, $\int_\Omega \rho \, \mathrm{d}V = 0$. The linearized equations around this equilibrium become

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\mathrm{div}\, \boldsymbol{u} \\ \rho \frac{\partial \boldsymbol{u}}{\partial t} &= \nu \Delta \boldsymbol{u} - b \nabla \rho. \end{aligned} \tag{6}$$

Hence the relevant semigroup, if any, must be generated by the operator

$$A \begin{pmatrix} \rho \\ \boldsymbol{u} \end{pmatrix} = \begin{pmatrix} -\mathrm{div}\, \boldsymbol{u} \\ \nu \Delta \boldsymbol{u} - b \nabla \rho \end{pmatrix}.$$

The spaces where A is defined depend on the problem. We will first deal with the case where Ω is a periodic box. Then we take for notational simplicity $\Omega = (0, 2\pi)^N$, where N is the space dimension. Let

$$\begin{aligned} H &= \left\{ (\rho, \boldsymbol{u}) \in L^2(\Omega)^{N+1} : \int_\Omega \rho \, \mathrm{d}V = 0, \int_\Omega \boldsymbol{u} \, \mathrm{d}V = \mathbf{0} \right\} \\ D(A) &= \left\{ (\rho, \boldsymbol{u}) \in L^2(\Omega) \times H^1(\Omega)^N : \right. \\ &\qquad \left. \int_\Omega \rho \, \mathrm{d}V = 0, \int_\Omega \boldsymbol{u} \, \mathrm{d}V = \mathbf{0}, \boldsymbol{u} \, \text{periodic}, \ \nu \Delta \boldsymbol{u} - b \nabla \rho \in L^2(\Omega)^N \right\}. \end{aligned} \tag{7}$$

The periodicity condition makes sense because in $H^1(\Omega)$ the traces at the boundary are well defined. In fact, if we just assume that $\boldsymbol{u} \in L^2(\Omega)^N$, and $\mathrm{div}\, \boldsymbol{u}$ (in the sense of distributions) lies within $L^2(\Omega)^N$, the normal component $\boldsymbol{u} \cdot \boldsymbol{n}$ makes sense [13, pp 237–251]. The value $\nu \Delta \boldsymbol{u} - b \nabla \rho$ is to be understood in the sense of distributions. Note that since the divergence of any periodic function within H^1 has zero mean, A takes $D(A)$ into H. The space of the velocities $\boldsymbol{u}$ satisfying the conditions of $D(A)$ is often denoted by $H^1_{\mathrm{per}}(\Omega)$, and its dual by $H^{-1}_{\mathrm{per}}(\Omega)$.

For the Dirichlet problem in a smooth domain Ω, the main spaces will be

$$\begin{aligned} H &= \left\{ (\rho, \boldsymbol{u}) \in L^2(\Omega)^{N+1} : \int_\Omega \rho \, \mathrm{d}V = 0 \right\} \\ D(A) &= \left\{ (\rho, \boldsymbol{u}) \in L^2(\Omega) \times H^1_0(\Omega)^N : \int_\Omega \rho \, \mathrm{d}V = 0, \nu \Delta \boldsymbol{u} - b \nabla \rho \in L^2(\Omega)^N \right\}. \end{aligned} \tag{8}$$

Note that since $\nabla \rho$ does not need to belong to $L^2(\Omega)^N$, then $(\rho, \boldsymbol{u}) \in D(A)$ does not necessarily imply $\boldsymbol{u} \in H^2(\Omega)^N$.

Let us see that A is a closed operator. If $(\rho_n, \boldsymbol{u}_n) \to (\rho, \boldsymbol{u})$ in H, $(-\mathrm{div}\, \boldsymbol{u}_n, \nu \Delta \boldsymbol{u}_n - b \nabla \rho_n) \to (f, \boldsymbol{g})$ in H, $b \nabla \rho_n + \boldsymbol{g}$ tends to $b \nabla \rho + \boldsymbol{g}$ in, respectively, $H^{-1}_{\mathrm{per}}(\Omega)$, $H^{-1}(\Omega)$. Since Δ is an isomorphism, $\Delta : H^1_{\mathrm{per}}(\Omega) \to H^{-1}_{\mathrm{per}}(\Omega)$ in the periodic case, $\Delta : H^1_0(\Omega) \to H^{-1}(\Omega)$ in the Dirichlet case, $\boldsymbol{u}_n$ tends to $\boldsymbol{u}$ not only in $L^2(\Omega)^N$, but also in $H^1(\Omega)^N$. Thus $\mathrm{div}\, \boldsymbol{u}_n$ tends to $\mathrm{div}\, \boldsymbol{u} = f$ in $L^2(\Omega)$, and since $\nu \Delta \boldsymbol{u} - b \nabla \rho = \boldsymbol{g} \in L^2(\Omega)^N$, $(\rho, \boldsymbol{u}) \in D(A)$.

3. The periodic problem

Recall that the Fourier transform takes H to

$$\hat{H} = \left\{ (\rho_{\boldsymbol{k}}, \boldsymbol{u}_{\boldsymbol{k}})_{\boldsymbol{k} \in \mathbb{Z}^N} : \rho_{\mathbf{0}} = 0, \boldsymbol{u}_{\mathbf{0}} = \mathbf{0}, \sum_{\boldsymbol{k}} |\rho_{\boldsymbol{k}}|^2 < \infty, \sum_{\boldsymbol{k}} |\boldsymbol{u}_{\boldsymbol{k}}|^2 < \infty \right\}$$

whereas, if $\boldsymbol{u} \in H^1_{\mathrm{per}}(\Omega)^N$, then $\sum k^2 |\boldsymbol{u}_{\boldsymbol{k}}|^2$ is also less than ∞.

Theorem 3.1. *The eigenvalues of A are the following ones:*

$$\{-\nu k^2, k = 1, 2, \ldots\}$$
$$\{\tfrac{1}{2}(-\nu k^2 - (\nu^2 k^4 - 4bk^2)^{1/2}), k = 1, 2, \ldots\}$$
$$\{\tfrac{1}{2}(-\nu k^2 + (\nu^2 k^4 - 4bk^2)^{1/2}), k = 1, 2, \ldots\}.$$

They have finite multiplicity. The first and second families accumulate at $-\infty$ when $k \to \infty$, whereas the third one tends to $-b/\nu$. All of them have negative real parts, and only a finite number are not real.

Proof. The Fourier transform diagonalizes A. Explicitly, the action of $\mathcal{F}^{-1} A \mathcal{F}$ upon a sequence $(\rho_{\boldsymbol{k}}, \boldsymbol{u}_{\boldsymbol{k}})$ takes its $\boldsymbol{k}$th component to

$$A_{\boldsymbol{k}} \begin{pmatrix} \rho_{\boldsymbol{k}} \\ u_{1,\boldsymbol{k}} \\ u_{2,\boldsymbol{k}} \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{i}k_1 & -\mathrm{i}k_2 \\ -\mathrm{i}bk_1 & -\nu k^2 & 0 \\ -\mathrm{i}bk_2 & 0 & -\nu k^2 \end{pmatrix} \begin{pmatrix} \rho_{\boldsymbol{k}} \\ u_{1,\boldsymbol{k}} \\ u_{2,\boldsymbol{k}} \end{pmatrix}$$

where for simplicity we have taken $N = 2$; the form of the matrix for dimension N follows the same scheme. The eigenvalues of $A_{\boldsymbol{k}}$ are $-\nu k^2$, with multiplicity $N - 2$ in the general case, and $(1/2)(-\nu k^2 \pm (\nu^2 k^4 - 4bk^2)^{1/2})$; they depend only of the value k^2. The properties of the families previously stated are simple calculations. □

Remark 1. All the nonreal eigenvalues lie within the curve $y^2 = x^2 + (2b/\nu)x$, $-2b/\nu < x < 0$.

Remark 2. The eigenvalues accumulating at $-\infty$ have an order $-\nu k^2$, and correspond to eigenvectors where $\rho_{\boldsymbol{k}}$ behaves like $\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{u}_{\boldsymbol{k}}/k^2$, so that $|\rho_{\boldsymbol{k}}| \ll |\boldsymbol{u}_{\boldsymbol{k}}|$ when $k \to \infty$. In a sense they are associated to incompressible oscillations of the velocity, which agrees with the fact that they are also present in the incompressible case. However, for the eigenvalues accumulating at $-b/\nu$, the eigenvectors satisfy $\boldsymbol{u}_{\boldsymbol{k}} \sim \mathrm{i}\boldsymbol{k}b\rho/k^2$, so that $|\boldsymbol{u}_{\boldsymbol{k}}| \ll |\rho_{\boldsymbol{k}}|$: they correspond to density oscillations without perturbation of the velocity. This is illuminating in a number of ways: for the incompressible case there are no finite accumulation points, which is natural in view of the above description; the fact that not all the spectrum is discrete means that $T(t)$ is not compact for any t. The full nonlinear Navier–Stokes equations present existence problems because of the noncompactness of certain approximate families of solutions, which is related to the phenomenon of persistent density oscillations, even for vanishing velocity [14, p 10]. We see that the linear approach already hints at some of the problems of the nonlinear theory.

Theorem 3.2. *The spectrum of A is formed by its eigenvalues plus the point $-b/\nu$.*

Proof. We will calculate the inverse of $A_{\boldsymbol{k}} - \lambda$ when λ is not an eigenvalue. Let

$$\begin{aligned} Q_k(\lambda) = -\det(A_{\boldsymbol{k}} - \lambda) &= (\lambda^2 + \nu\lambda k^2 + bk^2)(\lambda + \nu k^2) \\ &= (\lambda - \tfrac{1}{2}[-\nu k^2 - (\nu^2 k^4 - 4bk^2)^{1/2}]) \\ &\quad \times (\lambda - \tfrac{1}{2}[-\nu k^2 + (\nu^2 k^4 - 4bk^2)^{1/2}])(\lambda + \nu k^2). \end{aligned}$$

Then $(A_{\boldsymbol{k}} - \lambda)^{-1}$ is

$$\frac{1}{Q_k(\lambda)} \begin{pmatrix} -(\lambda + \nu k^2)^2 & \mathrm{i}k_1(\lambda + \nu k^2) & \mathrm{i}k_2(\lambda + \nu k^2) \\ \mathrm{i}bk_1(\lambda + \nu k^2) & -(\lambda^2 + (\lambda\nu + b)k^2 - bk_1^2) & bk_1k_2 \\ \mathrm{i}bk_2(\lambda + \nu k^2) & bk_1k_2 & -(\lambda^2 + (\lambda\nu + b)k^2 - bk_2^2) \end{pmatrix}$$

and the analogous matrix for dimension N. For λ at a positive distance from the set of eigenvalues, all the terms are uniformly bounded in k. In fact, for k large, after bounding below the only possible small factor $|\lambda-(1/2)[-\nu k^2+(\nu^2k^4-4bk^2)^{1/2}]|>r>0$, we are left with fractions of order k^2/k^2, in the (1, 1) position; k^2/k^4, in the (2, 3) and (3, 2) positions; and $1/k$ in the remaining positions. Thus the matrices $(A_{\boldsymbol{k}}-\lambda)^{-1}$ are uniformly bounded in $\boldsymbol{k}$ and the diagonal operator $(\mathcal{F}^{-1}A\mathcal{F}-\lambda)^{-1}$ is bounded. □

Inspection of $(A_{\boldsymbol{k}}-\lambda)^{-1}$ yields the following corollaries.

Corollary 3.3. *For $\lambda>0$, the family $(A_{\boldsymbol{k}}-\lambda)^{-1}$ takes square-summable sequences $(\rho_{\boldsymbol{k}},\boldsymbol{u}_{\boldsymbol{k}})$ into sequences $(\rho'_{\boldsymbol{k}},\boldsymbol{u}'_{\boldsymbol{k}})$ with $\sum|\rho'_{\boldsymbol{k}}|^2<\infty$, $\sum k^2|\boldsymbol{u}_{\boldsymbol{k}}|^2<\infty$.*

Proof. Except for the term (1, 1), all the remaining entries in $(A_{\boldsymbol{k}}-\lambda)^{-1}$ mostly behave like $1/k$, and the components of the velocity start at the second coordinate. □

Corollary 3.4. *The norms $\|(A_{\boldsymbol{k}}-\mathrm{i}r)^{-1}\|$ tend to zero uniformly in $\boldsymbol{k}$ when $r\to\pm\infty$.*

Proof. All the terms satisfy a bound of the form $M/|r|$, M independent of $\boldsymbol{k}$. □

The consequences of the above proof are as follows.

Inspection of the spectrum shows that it is located not only in the left half plane (which is obvious because the operator is dissipative) but also at a positive distance of $\mathrm{i}\mathbb{R}$.

Corollary (3.3) guarantees that for $\lambda>0$, $(A-\lambda)^{-1}$ takes H into $L^2(\Omega)\times H^1_{\mathrm{per}}(\Omega)^N$, and since the condition upon $\nu\Delta\boldsymbol{u}-b\nabla\rho$ is implicit in the definition of A, $\mathrm{Ran}(A-\lambda)=H$. Hence, by Lumer–Phillips' theorem [1, p 83] and [15], A generates a contraction semigroup, which takes care of condition (b).

Corollary (3.4) asserts that $\|(A\pm\mathrm{i}r)^{-1}\|\to0$ when $r\to\infty$, i.e. the semigroup $T(t)$ is immediately norm-continuous [1, p 115] (continuous in norm for $t>0$), so that $\sigma(T(t))-\{0\}=\mathrm{e}^{t\sigma(A)}$ [1, p 280], which yields condition (c) as mentioned in the introduction.

Finally, applying the dissipativity of the spectrum and the last remark, we find that $\sigma(T(1))$ is located in a ball of radius less than one, so the semigroup is almost-stable and condition (d) is proved.

4. The Dirichlet problem: general properties

Most of the following arguments work as well for the periodic case, but naturally the Fourier transform locates the spectrum of A more precisely. We will define in H an internal product, equivalent to the usual one, but more suited to our particular problem:

$$\langle(\rho_1,\boldsymbol{u}_1),(\rho_2,\boldsymbol{u}_2)\rangle=b(\rho_1,\rho_2)_{L^2}+(\boldsymbol{u}_1,\boldsymbol{u}_2)_{L^2}. \tag{9}$$

From now on an overbar will denote the complex conjugate.

Proposition 4.1. *The operator A is dissipative.*

Proof.

$$\begin{aligned}\langle A(\rho,\boldsymbol{u}),(\rho,\boldsymbol{u})\rangle&=-b\int_\Omega\bar\rho\,\mathrm{div}\,\boldsymbol{u}\,\mathrm{d}V+\int_\Omega\nu\Delta\boldsymbol{u}\cdot\bar{\boldsymbol{u}}\,\mathrm{d}V-b\int_\Omega\nabla\rho\cdot\bar{\boldsymbol{u}}\,\mathrm{d}V\\&=-\nu\int_\Omega|\nabla\boldsymbol{u}|^2\,\mathrm{d}V+2\mathrm{i}b\,\mathrm{Im}\left(\int_\Omega\rho\,\mathrm{div}\,\bar{\boldsymbol{u}}\,\mathrm{d}V\right).\end{aligned}$$

Hence

$$\mathrm{Re}\ (\langle A(\rho,\boldsymbol{u}),(\rho,\boldsymbol{u})\rangle)=-\nu\int_\Omega|\nabla\boldsymbol{u}|^2\leqslant0.$$

For the inviscid case, the operator iA becomes self-adjoint for this internal product, which without further ado takes care of properties (b)–(d). □

Theorem 4.2. *For* $\lambda > 0$ *large enough,* $\mathrm{Ran}\,(A-\lambda) = H$.

Proof. Let $(A-\lambda)(\rho, \boldsymbol{u}) = (\rho_1, \boldsymbol{u}_1)$. This means that

$$-\mathrm{div}\,\boldsymbol{u} - \lambda\rho = \rho_1$$
$$\nu\Delta\boldsymbol{u} - b\nabla\rho - \lambda\boldsymbol{u} = \boldsymbol{u}_1.$$

Formally,

$$-(b\,\mathrm{div}\,(\nu\Delta-\lambda)^{-1}\nabla + \lambda)\rho = \rho_1 + \mathrm{div}\,(\nu\Delta-\lambda)^{-1}\boldsymbol{u}_1$$
$$(\nu\Delta-\lambda)\boldsymbol{u} = \boldsymbol{u}_1 + b\nabla\rho.$$

Let us remember that for any $\lambda > 0$, $(\nu\Delta-\lambda)^{-1}$ is an isomorphism between the following spaces: $H^{-1}(\Omega)^N \to H_0^1(\Omega)^N$, $L^2(\Omega)^N \to H^2(\Omega)^N \cap H_0^1(\Omega)^N$. Moreover, its norm is bounded independently of λ when $\lambda \to \infty$. Now, since both ∇ and div take any space H^m into H^{m-1} in a continuous way, $b\,\mathrm{div}\,(\nu\Delta-\lambda)^{-1}\nabla$ takes H^m into H^m continuously, with a norm bounded independently of λ. Therefore, for λ larger than this common bound, $b\,\mathrm{div}\,(\nu\Delta-\lambda)^{-1}\nabla + \lambda$ is invertible among these spaces. Therefore, if $(\rho_1, \boldsymbol{u}_1) \in H$, since the term $\rho_1 + \mathrm{div}\,(\nu\Delta-\lambda)^{-1}\boldsymbol{u}_1 \in L^2(\Omega)$, ρ (given by the expression above) also lies within $L^2(\Omega)$. Moreover, the integral of ρ_1 is zero, and the divergence of any function of H_0^1 (such as $(\nu\Delta-\lambda)^{-1}\boldsymbol{u}_1$) also has integral zero; the same happens to ρ. Finally, $\boldsymbol{u} = (\nu\Delta-\lambda)^{-1}(\boldsymbol{u}_1 + b\nabla\rho)$ belongs to $H_0^1(\Omega)^N$ and $\nu\Delta\boldsymbol{u} - b\nabla\rho = \lambda\boldsymbol{u} + \boldsymbol{u}_1 \in L^2(\Omega)^N$. Thus the formal expression above really yields an antecedent within $D(A)$ of $(\rho_1, \boldsymbol{u}_1)$. In combination with (4.1) and Lumer–Phillips' theorem, we obtain the following corollary. □

Corollary 4.3. *A generates a contraction semigroup.*

Proposition 4.4. $(A \pm \mathrm{i}r)^{-1}$ *tends to zero in* $\mathcal{L}(H)$ *when* $r \to \infty$.

Proof. We have

$$\left|2b\,\mathrm{Im}\left(\int_\Omega \rho\,\mathrm{div}\,\bar{\boldsymbol{u}}\,\mathrm{d}V\right)\right| \leqslant 2b\,\|\mathrm{div}\,\boldsymbol{u}\|\,\|\rho\|$$
$$\leqslant b\varepsilon^2\,\|\mathrm{div}\,\boldsymbol{u}\|^2 + \frac{b}{\varepsilon^2}\|\rho\|^2 \leqslant b\varepsilon^2\|\nabla\boldsymbol{u}\|^2 + \frac{b}{\varepsilon^2}\|\rho\|^2$$

for any $\varepsilon > 0$. Take $\varepsilon^2 = \nu/2b$. Then

$$\left|2b\mathrm{Im}\left(\int_\Omega \rho\,\mathrm{div}\,\bar{\boldsymbol{u}}\right)\right| \leqslant \frac{\nu}{2}\|\nabla\boldsymbol{u}\|^2 + \frac{2b^2}{\nu}\|\rho\|^2.$$

Since

$$\langle (A \pm \mathrm{i}r)(\rho, \boldsymbol{u}), (\rho, \boldsymbol{u})\rangle = -\nu\int_\Omega |\nabla\boldsymbol{u}|^2\,\mathrm{d}V + 2\mathrm{i}b\,\mathrm{Im}\left(\int_\Omega \rho\,\mathrm{div}\,\bar{\boldsymbol{u}}\,\mathrm{d}V\right) \pm \mathrm{i}r\|(\rho, \boldsymbol{u})\|^2$$

we have

$$|\langle (A \pm \mathrm{i}r)(\rho, \boldsymbol{u}), (\rho, \boldsymbol{u})\rangle| \geqslant |-\nu\|\nabla\boldsymbol{u}\|^2 \pm \mathrm{i}r\|(\rho, \boldsymbol{u})\|^2| - \left|2b\,\mathrm{Im}\left(\int_\Omega \rho\,\mathrm{div}\;\bar{\boldsymbol{u}}\,\mathrm{d}V\right)\right|$$
$$\geqslant \left(\frac{\nu}{\sqrt{2}} - \frac{\nu}{2}\right)\|\nabla\boldsymbol{u}\|^2 + \frac{r}{\sqrt{2}}\|(\rho, \boldsymbol{u})\|^2 - \frac{2b^2}{\nu}\|\rho\|^2$$

and therefore

$$\|(A \pm \mathrm{i}r)(\rho, \boldsymbol{u})\| \geqslant \left(\frac{r}{\sqrt{2}} - \frac{2b^2}{\nu}\right) \|(\rho, \boldsymbol{u})\|$$

which implies the result. □

Corollary 4.5. *A generates an immediately continuous semigroup.*

Hence, properties (b), (c) hold for A. To see that condition (d) is also true, we need to show that $\sigma(A)$ lies in some half-plane to the left of the imaginary axis. This will follow from a study of the distribution of the spectrum of A, which in view of (c) is interesting in itself.

Corollary 4.6. *As an operator of $\mathcal{L}(L^2(\Omega)^{N+1})$, $(A-\lambda)^{-1}$ is never compact.*

Proof. We showed in theorem (4.2) that for $\lambda > 0$ large enough, the first component of $(A-\lambda)^{-1}(\rho_1, \mathbf{0})$ is $\rho = -(b\,\mathrm{div}\,(\nu\Delta - \lambda)^{-1}\nabla + \lambda)^{-1}\rho_1$. As stated, $\mathrm{div}\,(\nu\Delta-\lambda)^{-1}\nabla$ is a continuous operator from L^2 to L^2. The inverse of a continuous operator is never compact between infinite-dimensional spaces. For other values μ in the resolvent set, since

$$(A-\lambda)^{-1} = [(\lambda-\mu)(A-\lambda)^{-1} + I](A-\mu)^{-1}$$

$(A-\mu)^{-1}$ cannot be compact. □

Note that the weak point in compactness properties is always the presence of a variable density.

5. The Dirichlet problem: location of the spectrum

Theorem 5.1. $0 \notin \sigma(A)$.

Proof. In the first place, 0 is not an eigenvalue of A: since $\mathrm{Re}\,\langle A(\rho, \boldsymbol{u}), (\rho, \boldsymbol{u})\rangle = -\nu\|\nabla\boldsymbol{u}\|^2$, $A(\rho, \boldsymbol{u}) = (0, \mathbf{0})$ would imply that, since $\boldsymbol{u} \in H_0^1(\Omega)^N$, $\boldsymbol{u} = \mathbf{0}$. Since $A(\rho, \mathbf{0}) = (0, -b\nabla\rho)$ and the mean of ρ is zero, ρ is also zero.

Second, 0 does not lie within the residual spectrum of A. In that case it would be an eigenvalue of A^*. However, A^* is obtained by changing the sign of the first-order derivatives, so that if $A^*(\rho, \boldsymbol{u}) = (0, \mathbf{0})$, $A(-\rho, \boldsymbol{u}) = (0, \mathbf{0})$.

It remains to see that 0 does not belong to the approximate spectrum of A. If so, there would be a sequence $(\rho_n, \boldsymbol{u}_n) \in D(A)$, with L^2-norm one, such that $A(\rho_n, \boldsymbol{u}_n) \to (0, \mathbf{0})$ in L^2. Since $\mathrm{Re}\,\langle A(\rho_n, \boldsymbol{u}_n), (\rho_n, \boldsymbol{u}_n)\rangle = -\nu\|\nabla\boldsymbol{u}_n\|^2$, necessarily $\boldsymbol{u}_n \to \mathbf{0}$ in H_0^1. Now take the unique $\phi_n \in H^2(\Omega)^N \cap H_0^1(\Omega)^N$ such that $\Delta\phi_n = \rho_n$. There is a constant M such that $\|\nabla\phi_n\|_{H^1} \leqslant M\|\rho_n\| \leqslant M$. Since $\nu\Delta\boldsymbol{u}_n - b\nabla\rho_n \to \mathbf{0}$ in L^2, $(\nu\Delta\boldsymbol{u}_n, \nabla\phi_n) - b(\nabla\rho_n, \nabla\phi_n) \to 0$. Now, this product is $-(\nu\nabla\boldsymbol{u}_n, \nabla(\nabla\phi_n)) + b(\rho_n, \mathrm{div}\,\nabla\phi_n) \to 0$. Since $\|\nabla\phi_n\|_{H^1} \leqslant M$, $\|\nabla(\nabla\phi_n)\|_{L^2} \leqslant M$, and $\|\nabla\boldsymbol{u}_n\|_{L^2} \to 0$, the first term tends to zero. The second is $b\|\rho_n\|^2$, which must also tend to zero, contrary to the initial condition. □

Theorem 5.2. *Except for the point $-b/\nu$, the spectrum of A is formed by eigenvalues of finite multiplicity.*

Proof. Let $\lambda \neq 0$, $(A-\lambda)(\rho, \boldsymbol{u}) = (\rho_1, \boldsymbol{u}_1)$. That means

$$\rho = -\frac{1}{\lambda}(\operatorname{div} \boldsymbol{u} + \rho_1) \tag{10}$$

and therefore

$$\nu\Delta\boldsymbol{u} + \frac{b}{\lambda}\nabla\operatorname{div}\boldsymbol{u} - \lambda\boldsymbol{u} = \boldsymbol{u}_1 - \frac{b}{\lambda}\nabla\rho_1. \tag{11}$$

Let us consider the differential operator $D_\lambda : H_0^1(\Omega)^N \to H^{-1}(\Omega)^N$:

$$D_\lambda\boldsymbol{u} = \nu\Delta\boldsymbol{u} + \frac{b}{\lambda}\nabla\operatorname{div}\boldsymbol{u}. \tag{12}$$

By substituting ∂_i by ξ_i, we obtain the self-adjoint polynomial matrix (written for simplicity for $N = 2$)

$$\nu\xi^2 I + \frac{b}{\lambda}\begin{pmatrix} \xi_1^2 & \xi_1\xi_2 \\ \xi_1\xi_2 & \xi_2^2 \end{pmatrix} = \nu\xi^2 I + \frac{b}{\lambda}L. \tag{13}$$

The eigenvalues of L are $\xi^2 = \xi_1^2+\xi_2^2$, 0 ($\xi^2, 0, \ldots, 0$ for dimension N). Hence the eigenvalues of the whole matrix are $(\nu + b/\lambda)\xi^2$, $\nu\xi^2, \ldots, \nu\xi^2$. None of them vanishes for any $\xi \neq 0$ except for $\lambda = -b/\nu$. Otherwise D_λ is elliptic (strongly elliptic for Re $(b/\lambda) > -\nu$, i.e. (Re $\lambda)/(|\lambda|^2) > -\nu/b$. Thus for $\lambda \neq -b/\nu$ the spectrum of D_λ is formed by a countable number of eigenvalues of finite multiplicity. If λ is one of them, there exists a solution to $D_\lambda\boldsymbol{u} = \boldsymbol{u}$, and it lies within $H_0^1(\Omega)^N \cap H^2(\Omega)^N$. Then the pair $(-\operatorname{div}\boldsymbol{u}, \boldsymbol{u})$ is an eigenvector for A with eigenvalue λ. On the other hand, if $D_\lambda - \lambda$ is invertible, it takes $L^2(\Omega)^N$ to $H_0^1(\Omega)^N \cap H^2(\Omega)^N$ continuously. By defining

$$\begin{aligned} \rho &= -\frac{1}{\lambda}\left(\operatorname{div}(D_\lambda - \lambda)^{-1}\left(\boldsymbol{u}_1 - \frac{b}{\lambda}\nabla\rho_1\right) + \rho_1\right) \\ \boldsymbol{u} &= (D_\lambda - \lambda)^{-1}\left(\boldsymbol{u}_1 - \frac{b}{\lambda}\nabla\rho_1\right) \end{aligned} \tag{14}$$

we obtain a continuous inverse of $A - \lambda$ taking H to $D(A)$.

Hence λ is a spectral point of A if and only if λ is an eigenvalue of D_λ, and in that case it is an eigenvalue of A. The relation between eigenvectors is $\boldsymbol{u} \to (\operatorname{div}\boldsymbol{u}, \boldsymbol{u})$.

Now the eigenvalues of D_λ are the zeros of an integral function $E(\lambda, z)$ called the discriminant; the analytic dependence of D_λ on λ makes sure that E is also an analytic function of λ, defined in $\mathbb{C} - \{0, -b/\nu\}$ [16, 17]. Hence the spectrum of A is formed by the zeros of $E(\lambda, \lambda)$; this analytic function does not vanish identically because $\sigma(A) \neq \mathbb{C}$. Thus it has a countable number of zeros. They do not accumulate at zero because this value does not belong to the spectrum of A; they can only accumulate at $-b/\nu$. By the arguments before, they are all eigenvalues of finite multiplicity. □

Proposition 5.3. *$-b/\nu$ is an accumulation point of eigenvalues.*

Proof. Although a completely detailed argument needs some perturbation theorems [17], the essence of the proof is as follows: for large eigenvalues λ, according to equations (8), (9), any eigenvector of A tends to have the form $(0, \boldsymbol{e}_n)$, where $\boldsymbol{e}_n$ is an eigenvector of $\nu\Delta$ within $H_0^1(\Omega)^N$ with eigenvalue λ_n. Now, those eigenvectors form an orthonormal basis of $L^2(\Omega)^N$ and for any $r > 0$, the family $1/(\lambda_n - r)$ is p-summable for any $p > N/2$. If we assume that $\sigma(A)$ is a discrete set, except for a finite number of eigenvalues, the eigenvectors of $(A - r)^{-1}$ tend to be the orthonormal family $(0, \boldsymbol{e}_n)$, and the eigenvalues $1/(\lambda_n - r)$ are p-summable;

thus $(A-r)^{-1}$ would belong to the Von Neumann–Schatten class $\mathcal{S}_p$ [16], and therefore would be a compact operator, contrary to the result proved in corollary 4.6. $\square$

There is a decomposition argument which highlights the similarities and differences with the periodic case. It is known (see e.g. [13]) that the space $L^2(\Omega)^N$ admits an orthogonal decomposition

$$L^2(\Omega)^N = H_0(\operatorname{div} 0, \Omega) \oplus \nabla H^1(\Omega)$$

where $H_0(\operatorname{div} 0, \Omega)$ is the space of square-integrable functions with null divergence (in the sense of distributions) such that the trace $\boldsymbol{u} \cdot \boldsymbol{n}$, which as stated before makes sense for those functions, vanishes at the boundary. Thus every $\boldsymbol{u} \in L^2(\Omega)^N$ may be decomposed

$$\begin{aligned} &\boldsymbol{u} = \boldsymbol{u}_0 + \nabla\phi \\ &\operatorname{div} \boldsymbol{u}_0 = 0 \\ &\boldsymbol{u}_0 \cdot \boldsymbol{n}|_{\partial\Omega} = 0 \\ &\phi \in H^1(\Omega). \end{aligned}$$

Of course ϕ is the solution of

$$\begin{aligned} &\Delta\phi = \operatorname{div} \boldsymbol{u} \\ &\left.\frac{\partial\phi}{\partial n}\right|_{\partial\Omega} = 0. \end{aligned}$$

ϕ is determined up to an additive constant, and $\nabla\phi$ uniquely: if we impose $\int_\Omega \phi \,\mathrm{d}V = 0$, ϕ is also unique. $\boldsymbol{u}_0$ is merely $\boldsymbol{u} - \nabla\phi$.

The equation $-\operatorname{div} \boldsymbol{u} = \lambda\rho$ becomes $\Delta\phi = -\lambda\rho$ that, when plugged into $\nu\Delta\boldsymbol{u} - b\nabla\rho = \lambda\boldsymbol{u}$, yields

$$\nu\Delta\boldsymbol{u}_0 - \lambda\boldsymbol{u}_0 = -\nabla\left(\nu + \frac{b}{\lambda}\Delta\phi + \lambda\phi\right). \tag{15}$$

The left-hand term is divergence free, and the right-hand one is a gradient. Hence this must be the gradient of some harmonic function ψ. Thus ϕ must satisfy

$$\begin{aligned} &\Delta\phi + \frac{\lambda^2}{\nu\lambda + b}\phi = -\frac{\lambda}{\nu\lambda + b}\psi \\ &\left.\frac{\partial\phi}{\partial n}\right|_{\partial\Omega} = 0 \\ &\int_\Omega \phi \,\mathrm{d}V = 0. \end{aligned} \tag{16}$$

This solution will have at the boundary some tangential values $\nabla\phi \times \boldsymbol{n}$. Then $\boldsymbol{u}_0$ must satisfy

$$\begin{aligned} &(\nu\Delta - \lambda)\boldsymbol{u}_0 = \nabla\psi \\ &\boldsymbol{u}_0 \cdot \boldsymbol{n}|_{\partial\Omega} = 0 \\ &\boldsymbol{u}_0 \times \boldsymbol{n}|_{\partial\Omega} = -\nabla\phi \times \boldsymbol{n}|_{\partial\Omega}. \end{aligned} \tag{17}$$

The last condition being set such that $\boldsymbol{u}_0 + \nabla\phi \in H_0^1(\Omega)^N$.

The trouble with this construction is that the condition $\operatorname{div} \boldsymbol{u}_0 = 0$ does not follow from the last equation. If, however, ψ and λ may be found such that this condition holds, λ is an eigenvalue of A associated to $\boldsymbol{u}_0 + \nabla\phi$. Conversely, every eigenvalue of A corresponds to some harmonic function ψ through this scheme.

Try $\psi = 0$. Then $\lambda^2/(\nu\lambda + b)$ must be an eigenvalue of the Laplacian for the Neumann condition. If we denote one of these eigenvalues by μ_n, $\lambda = (\mu_n \pm (\mu_n^2\nu_2 - 4\mu_n b)^{1/2})/2$.

Now this λ, unless it is an eigenvalue of $\nu\Delta$, will yield a unique solution of the inhomogeneous Dirichlet problem for $\boldsymbol{u}_0$. If this solution happens to be solenoidal, $\lambda \in \sigma(A)$.

This seems to be rather a strange coincidence, but it always happens in the periodic case. Then periodicity is automatically satisfied for the complex exponentials, and $\mu_n = -n^2$.

On the other hand, if λ is an eigenvalue of $\nu\Delta$ for the homogeneous Dirichlet problem with a solenoidal eigenfunction, we may always set $\phi = \psi = 0$, $\boldsymbol{u} = \boldsymbol{u}_0$, $\rho = 0$. Again this is always possible for the periodic case: just take $\boldsymbol{r}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$, $\boldsymbol{r}\cdot\boldsymbol{k} = 0$, $\lambda = -\nu k^2$. However, in general, not every eigenspace of Δ will posses a solenoidal function.

With those overabundant restrictions upon the eigenvalues of A, it seems difficult to find out much about them. We will see, however, that the nonreal ones satisfy a bound similar to the one in the periodic problem.

Proposition 5.4. *All the nonreal eigenvalues of A are located within the region*

$$\left\{(x, y) \in \mathbb{C} : 0 \leqslant y^2 \leqslant -\frac{2b}{\nu}x - x^2\right\}.$$

Proof. Since

$$\int_\Omega \nabla \mathrm{div}\, \boldsymbol{u} \cdot \bar{\boldsymbol{u}}\, \mathrm{d}V = -\int_\Omega |\mathrm{div}\, \boldsymbol{u}|^2\, \mathrm{d}V$$

for an eigenfunction $\boldsymbol{u}$ of D_λ with $\|\boldsymbol{u}\| = 1$, the internal product in $L^2(\Omega)^N$,

$$\langle D_\lambda \boldsymbol{u} - \lambda \boldsymbol{u}, \boldsymbol{u}\rangle = -\nu\|\nabla \boldsymbol{u}\|^2 - \frac{b}{\lambda}\|\mathrm{div}\, \boldsymbol{u}\|^2 - \lambda = 0. \tag{18}$$

By taking imaginary parts,

$$\mathrm{Im}\, \left(\frac{b\, \|\mathrm{div}\, \boldsymbol{u}\|^2}{|\lambda|^2} - 1\right) = 0. \tag{19}$$

For nonreal λ, therefore, $\|\mathrm{div}\, \boldsymbol{u}\|^2 = |\lambda|^2/b$. Now taking real parts,

$$-\nu\|\nabla \boldsymbol{u}\|^2 - \frac{b}{|\lambda|^2}(\mathrm{Re}\ \lambda)\|\mathrm{div}\, \boldsymbol{u}\|^2 - \mathrm{Re}\ \lambda = 0 \tag{20}$$

which means $2\mathrm{Re}\ \lambda = -\nu\|\nabla \boldsymbol{u}\|^2$. Since $\|\mathrm{div}\, \boldsymbol{u}\| \leqslant \|\nabla \boldsymbol{u}\|$,

$$\nu|\lambda|^2 \leqslant -2b\, \mathrm{Re}\ \lambda \leqslant 2b|\lambda| \tag{21}$$

which proves the proposition. □

Corollary 5.5. *Property (d) holds for the homogeneous Dirichlet problem.*

Proof. The compact region of (5.4) contacts the imaginary axis only at 0, which is not part of the spectrum. By excluding a neighbourhood around $z = 0$ without eigenvalues, we find that the spectrum lies within a half-plane Re $z \leqslant -\delta < 0$. Hence $|\mathrm{e}^z| \leqslant \mathrm{e}^{-\delta} < 1$ for all $z \in \sigma(A)$ and the semigroup is almost-stable.

Finally, since the coefficients of the equation are real, it is obvious that $\lambda \in \sigma(A)$ if and only if $\bar{\lambda} \in \sigma(A)$. Note that eigenvectors associated to nonreal eigenvalues satisfy $\|\nabla \boldsymbol{u}\|^2 = -2\mathrm{Re}\ \lambda/\nu < 4b/\nu^2$, these eigenvectors form a bounded family in $H_0^1(\Omega)^N$, and therefore a relatively compact set in $L^2(\Omega)^N$. □

6. Conclusions

We have studied the equations of Navier–Stokes for a viscous, barotropic compressible fluid, linearized around a static equilibrium. Our first aim is to show that the associated linear operator A generates a strongly continuous (or C_0) semigroup $T(t)$, that the spectrum of A yields all the necessary information to find $\sigma(T(t))$, and that the semigroup is almost-stable. Those conditions are essential for the spectral analysis of the linearized equations to have any reliability as a predictive tool. We prove that all these statements are true for the periodic and the homogeneous Dirichlet problems, in the first case by Fourier analysis and in the second by functional analytic techniques. In addition, the spectrum of the periodic operator is found exactly, showing that it is discrete except for a unique finite accumulation point. The same result is proved for the homogeneous Dirichlet problem by different methods, showing that the spectrum has qualitatively an analogous distribution.

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